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Proca Effect in Reissner–Nordstrom de Sitter Metric

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Reissner–Nordstrom de Sitter spacetime with photon rest mass is studied. An iteration method is used to get the metric of this spacetime. In the case of $\mu \to 0$, the solution will return to the common Reissner–Nordstrom de Sitter spacetime.

KEY WORDS: Reissner–Nordstrom de Sitter metric; photon rest mass; stress-energy tensor.

1. INTRODUCTION

The recent WMAP's data prefer a positive cosmological constant (Bennett *et al.*, 2003). It means that our cosmos is semble as the de Sitter spacetime. So people pay more attention to the research about the de Sitter spacetime. As early as in 1917, de Sitter gave the spacetime metric in the sphere symmetry vacuum field with a positive cosmic constant Λ. Recently someone put forward the field solution of the Reissner–Nordstrom de Sitter spacetime (Ghosh, 2004; Salgado, 2003)

$$d\tau^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

It is a metric of no photon's rest mass. Moreover it is an important problem whether a photon has rest mass or not. During the last three decades, the problem captured special attention of many investigators. Recently, Lakes (1998), and Luo *et al.* (2003) measured the rest mass of photon by means of rotating torsion balance, respectively. The most new upper limit on photon rest mass is 1.2×10^{-54} kg (Luo

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et al., 2003). However what is the Reissner–Nordstrom de Sitter spacetime when the photon contains a rest mass?

The Proca equation for photon is the natural extension of the Maxwell equation to case with the rest mass (Proca, 1936). The accurate solution of the Proca field equation of a point charge in the curve spacetime has been discussed (Goldhaber and Nieto, 1971; Kramer *et al.*, 1980). While studying Einstein–Proca system, various approximations were utilized in searching solutions of the system in more complicated cases. Numerical solutions were found by Obukov and Vlachynsky (2000) and Toussaint (1999) independently. Vuille *et al.* (2002) solved the Einstein–Proca field equations in point charge field when $\Lambda = 0$ by perturbation analysis. This article will discuss the Proca effect in the Reissner–Nordstrom de Sitter spacetime.

Considering the de Sitter spacetime far from the charged bodies, then $\frac{2M}{r} \ll 1$ (it has adopted the natural unit system here, the same to the following). In the ken of the cosmos $\Lambda r^2 \ll 1$ holds, because the horizon of the cosmos is 1.5×10^{10} light years and the estimated cosmic constant is about $\Lambda \sim 2.5 \times 10^{-55} \frac{1}{m^2}$ (Padmanahan, 2003; Carmeli, 2001). So the metric of the flat spacetime can be taken as the initial value in the iterative computation. After getting the solution of the Einstein–Proca equation, we substitute it into the Eistein field equation. The small quantity of high orders is abandoned in the calculation. Then the Reissner–Nordstrom de Sitter metric is obtained when photon has a nonzero rest mass. Moreover, the metric will return to the common Reissner–Nordstrom de Sitter spacetime when $\mu \to 0$.

2. THE SOLUTION OF THE FIELD EQUATION

The equation for a particle exhibiting a spin-1 short or intermediate-range field in flat space is Proca's equation (Proca, 1936), which in the absence of currents is

$$\partial_a F^{ab} + \mu^2 A^b = 0. \tag{1}$$

where

$$F_{ab} = \nabla_a A_b - \nabla_b A_a. \tag{2}$$

The Proca equation (1) can be expressed as the general form (Obukov and Vlachynsky, 2000) in the curve spacetime

$$\frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} F^{ab}) + \mu^2 A^b = 0. \tag{3}$$

where g is the determinant of the metric. The covariant derivative for the electromagnetic tensor is

$$F^{\mu\nu}_{;\nu} = \xi^\mu g_{00}^{-\frac{1}{2}} B^\nu_{;\nu}. \tag{4}$$

Here ξ^μ is timelike Killing vector, $g_{00} = \xi^\mu \xi_\mu = \frac{1}{|g|}$ and $B_\mu = (\xi^\mu \xi_\mu)^{-\frac{1}{2}} A_{0;\mu}$.

Applying this formula to Equation (3), we get

$$\begin{aligned}
&\varepsilon^\mu \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\nu}(\sqrt{|g|} g_{00}^{-1} g^{\sigma\nu} A_{0,\sigma}) + \mu^2 A^\mu = 0; \\
&\sigma = 1.2.3 \\
&\varepsilon^0 = 1 \\
&\varepsilon^i = 0.
\end{aligned} \tag{5}$$

Obviously, the scalar potential for the electromagnetic field of the electric charge is nonzero.

Firstly, we can substitute the Schwarzschild de Sitter metric $d\tau^2 = -(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2)dt^2 + (1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$ into the Equation (5), then we get

$$\begin{aligned}
\left(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\right)(r^2 A_{0;r})_{;r} + \frac{1}{\sin\theta}(\sin\theta A_{0;\theta})_{;\theta} \\
+ \frac{1}{\sin^2\theta} A_{0;\varphi\varphi} - (\mu \cdot r)^2 A_0 = 0.
\end{aligned} \tag{6}$$

The solution of the Equation (6) has the following general form

$$A_0(r, \theta) = R(r) p_l^n(\cos\theta) e^{in\varphi} \tag{7}$$

where

$$\left(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\right)(2rR_{,r} + r^2 R_{,rr}) - l(l+1)R - (\mu \cdot r)^2 R = 0. \tag{8}$$

$l = 0$ for the field of point electric charge, then

$$\left(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\right)(2rR_{,r} + r^2 R_{,rr}) - (\mu \cdot r)^2 R = 0. \tag{9}$$

For a large distance, $\frac{2M}{r} \ll 1$ and the estimated cosmic constant is about $\Lambda \sim 2.5 \times 10^{-55} \frac{1}{m^2}$, therefore $\Lambda r^2 \ll 1$ holds in the horizon of the universe. With the above-mentioned approximate condition, we can get the solution from Equation (9) as follows

$$R = \frac{Ce^{-ur}}{r}. \tag{10}$$

We have $C = Q$ in the natural unit system for the electric charge of a celestial body. Correspondingly, the electromagnetism vector potential is

$$A_i = \frac{Qe^{-\mu r}}{r} \delta_{0i}. \tag{11}$$

The electromagnetism stress-energy tensor of the photon mass is:

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b^c - \frac{1}{4}g_{ab}F_{cd}F^{cd} - \mu^2 A_a A_b + \frac{\mu^2}{2}g_{ab}A_c A^c\right). \tag{12}$$

We resolve the Equation (12) by taking the vector A_i (11) as the initial value of the vector potential. The electromagnetism stress-energy tensor can be represented as

$$\begin{aligned}
T_{00} &= \frac{1}{8\pi}\left[g^{11}\frac{Q^2 e^{-2\mu r}(1+\mu r)^2}{r^4} - \mu^2\frac{Q^2 e^{-2\mu r}}{r^2}\right]\\
T_{11} &= \frac{1}{8\pi}\left[g^{00}\frac{Q^2 e^{-2\mu r}(1+\mu r)^2}{r^4} + \mu^2\frac{Q^2 e^{-2\mu r}}{r^2}g_{11}g^{00}\right]\\
T_{22} &= \frac{1}{8\pi}\left[-g_{22}g^{00}g^{11}\frac{Q^2 e^{-2\mu r}(1+\mu r)^2}{r^4} + \mu^2\frac{Q^2 e^{-2\mu r}}{r^2}g_{22}g^{00}\right]\\
T_{33} &= T_{22}\cdot\sin^2\theta.
\end{aligned} \tag{13}$$

A nonzero photon rest mass can change the spacetime metric. So every metric must be revised when a photon has rest mass. The Reissner–Nordstrom de Sitter spacetime with a nonzero photon mass, therefore, can be expressed as

$$\begin{aligned}
d\tau^2 = &-\left(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2 + u\right)dt^2 + \left(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2 + v\right)^{-1}dr^2\\
&+ r^2(1+w)(d\theta^2 + \sin^2(\theta)\,d\varphi^2)
\end{aligned} \tag{14}$$

The metric contained the first order correction u, v and w. We can write the Equation (14) as a contracted form as

$$d\tau^2 = -Bdt^2 + Adr^2 + C(d\theta^2 + \sin^2\theta d\varphi^2). \tag{15}$$

For a point charge, the pending parameters u, v, w have nothing to do with θ, φ and the Riemann curve tensor can be expressed as follows

$$\begin{aligned}
R_{00} &= \frac{B''}{2A} - \frac{1}{4}\frac{B'}{A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{A}\frac{C'}{2C}\\
R_{11} &= -\frac{B''}{2B} + \frac{1}{4}\frac{B'}{B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A}{A}\frac{C'}{2C} + \frac{C'^2}{2C^2} - \frac{C''}{C}\\
R_{22} &= 1 + \frac{C'}{4A}\left(\frac{A'}{A} - \frac{B'}{B}\right) - \frac{C''}{2A}\\
R_{33} &= R_{22}\sin^2\theta
\end{aligned} \tag{16}$$

Maxwell stress-energy tensor is a traceless one. Therefore, Einstein field equation reads

$$R_{ab} = \Lambda g_{ab} + 8\pi T_{ab}. \tag{17}$$

Applying the electromagnetism stress-energy tensor (13) and the Riemann curve tensor (16) into the field Equation (17), we get

$$\begin{aligned}
&\frac{u''}{2} + \frac{u'}{r} = \frac{Q^2 e^{-2\mu r}(1 + 2\mu r)}{r^4} \\
&\frac{r}{2}v' + w' r + \frac{w''}{2} r^2 = \frac{Q^2 e^{-2\mu r}(\mu^2 r^2 - 1)}{r^2} \\
&\frac{r}{2}v' + v + 2rw' + \frac{r^2 w''}{2} = -\frac{2\mu Q^2 e^{-2\mu r}}{r}
\end{aligned} \tag{18}$$

The solutions of the equations are

$$\begin{aligned}
u &= \frac{Q^2}{r^2} e^{-2\mu r} - \frac{2\mu Q^2}{r} e^{-2\mu r} - 4u^2 Q^2 \int_r^\infty \frac{e^{-2\mu r}}{r} dr + \frac{c_1}{r} + c_2 \\
v &= \frac{Q^2 e^{-2\mu r}}{r^2} - \frac{2\mu Q^2}{r} e^{-2\mu r} + \mu^2 Q^2 e^{-2\mu r} + 2r\mu^3 Q^2 e^{-2\mu r} \\
w &= \mu^2 Q^2 e^{-2\mu r} + 2\mu^2 Q^2 \int_r^\infty \frac{e^{-2\mu r}}{r} dr + c_3.
\end{aligned} \tag{19}$$

Thus the Reissner–Nordstrom de Sitter spacetime with a nonzero photon mass is obtained. When $\mu = 0, \Lambda = 0$, we get $c_1 = c_2 = c_3 = 0$ by comparing the solution with the Schwarzschild outer solution. The metric of the Reissner–Nordstrom de Sitter spacetime with a nonzero photon mass is expressed as follows

$$\begin{aligned}
d\tau^2 = &-\Bigg(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2 + \frac{Q^2}{r^2} e^{-2\mu r} - \frac{2\mu Q^2}{r} e^{-2\varpi r} \\
&- 4\mu^2 Q^2 \int_r^\infty \frac{e^{-2\mu r}}{r} dr\Bigg) dt^2 + \Bigg(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2 + \frac{Q^2}{r^2} e^{-2\mu r} \\
&- \frac{2\mu Q^2}{r} e^{-2\mu r} + \mu^2 Q^2 e^{-2\mu r} + 2r\mu^3 Q^2 e^{-2\mu r}\Bigg)^{-1} dr^2 \\
&+ r^2 \left(1 + \mu^2 Q^2 e^{-2\mu r} + 2\mu^2 Q^2 \int_r^\infty \frac{e^{-2\mu r}}{r} dr\right) (d\theta^2 + \sin^2(\theta)\, d\varphi^2)
\end{aligned} \tag{20}$$

3. DISCUSSION

The Reisssner–Norderstrom de Sitter metric with a nonzero photon mass is given. It can return to the general Reissner–Nordstrom de Sitter metric (Ghosh, 2004; Salgado, 2003) when $\mu = 0$. When $\Lambda = 0$, the metric is different from that in Vuille *et al.* (2002), because a term of stress-energy tensor in proportion to μ^2 is ignored in their work.

When r is very large, the index in the Proca term becomes predominant. The terms of charge will be far smaller than the front terms. So, we still can take no account of the influence of the electric charge for the interaction of celestial bodies when the photon has a rest mass. Moreover, most of celestial bodies including black holes are rolling, so Kerr–Newman de Sitter metric needs to be researched ulteriorly.

REFERENCES

Bennett, C. L., Halpern, M., Hinshaw, G. et al. (2003). [astro-ph/0302207].
Carmeli, M. (2001) [astro-ph/0111259].
Ghosh, K. (2004) [gr-qc/0212060].
Goldhaber, A. S. and Nieto, M. M. (1971). *Reviews of Modern Physics* **43**, 277.
Lakes, R. (1998). *Physical Review Letters* **80**, 1826.
Luo, J., Tu, L. C., Hu, Z. K., and Luan, E. J. (2003). *Physical Review Letters* **90**, 081801.
Obukov, Y. and Vlachynsky, E. J. (2000). Preprint gr-qc/0004081 28 April 2000.
Padmanahan, T. (2003). *Physics Reports* **380**, 235–320.
Proca, A. (1936). *Le Journal de Physique et le Radium* **7**, 347.
Kramer, D., Stephani, H., Herlt, E., and MacCallum, M. (1980). *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, Cambridge.
Salgado, M. (2003) [gr-qc/0304010].
Toussaint, M. (1999). Preprint gr-qc/991042, 12 October 1999.
Vuille, C., Ipser, J., and Gallag, J. (2002). *General Relativity and Gravitation* **34**(5).